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Dynamics of the Vortex Lines Density in Superfluid Turbulence

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Dynamics of the Vortex Lines Density in Superfluid Turbulence

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Motivation

Self-consistent description of the superfluid channel flow

Outline

- Few facts about Superfluid He II
- Equation of Motion for VLD
- Dynamics of VLD in inhomogeneous flow
- Numerical Setup
- Results
- Summary

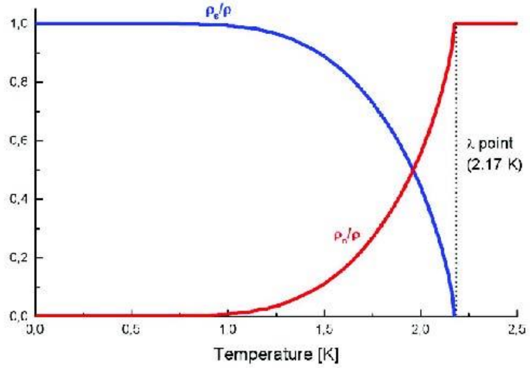
Few facts about Superfluid He II

- Following discovery of superfluidity in ^{4}He by Kapitza and Allen and Missener (1937) Tisza (1940) and Landau(1941) suggested a Two-fluid model :

  Liquid ^{4}He below T_c consists of two interpenetrating fluids – the normal fluid (with very low kinematic viscosity ν_n and density ρ_n) and the inviscid superfluid with density ρ_s,

  $\rho_n + \rho_s = \rho$ density of ^{4}He for T <2.17 K

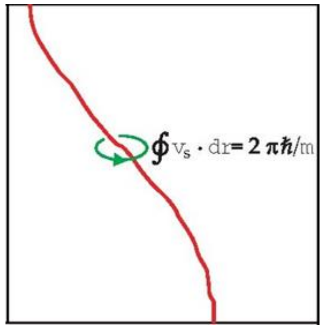
  $\nu_n = 10^{-4}\ cm^2$ /s at T=2 K



- In the superfluid, the vorticity is restricted to a set of vortex lines with circulation quantized to multiples of the circulation quantum

  $\kappa = h / m_{^4He}$

  for ^{4}He the core radius $a_0 \approx 10^{-8}\ cm; \kappa = 9.97 \times 10^{-4}\ cm^2$ /s .



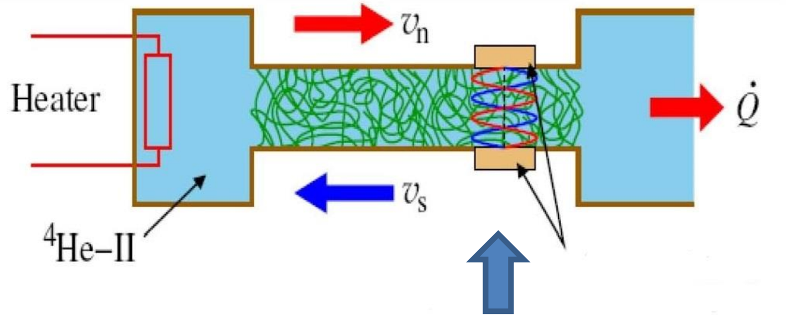
- The term “superfluid turbulence” refers to a chaotic tangle of interacting quantized vortex lines (R.P. Feinman, *Prog in Low Temp. Phys*., vol. 1(1955))

- The superfluid turbulence may be excited mechanically or by heat currents (counterflow turbulence).

Thermal Counterflow in He II

A form of motion unique to two-fluid superfluid hydrodynamics, no direct analogy in any ordinary viscous fluid.

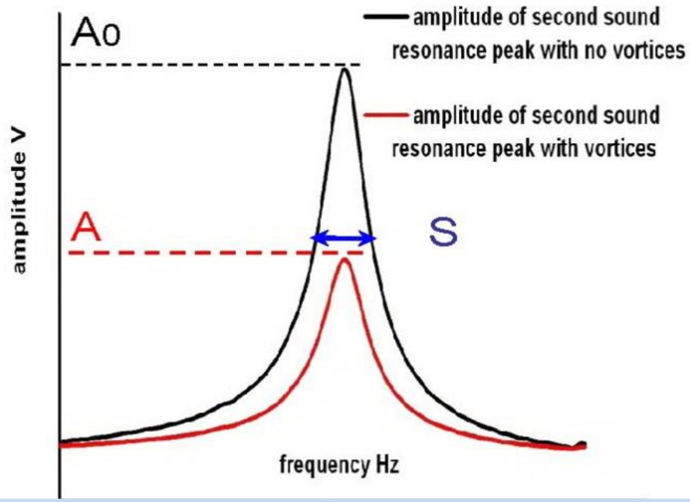
Thermal counterflow may be set up by applying a current to a heater located at the closed end of a channel open to a helium bath at the other end.



The heat flux $\dot{Q}$ is carried away from the heater by the normal fluid alone, and, by conservation of mass $\rho_n V_n + \rho_s V_s = 0$, a superfluid current arises in the opposite direction.

In this way a relative (counterflow) velocity $V_{ns} = V_n - V_s$ is created along the channel, which leads to appearance of a disordered tangle of superfluid vortex lines with density $\mathcal{L}$.

second sound (fluctuations of temperature) is attenuated by superfluid vortex lines



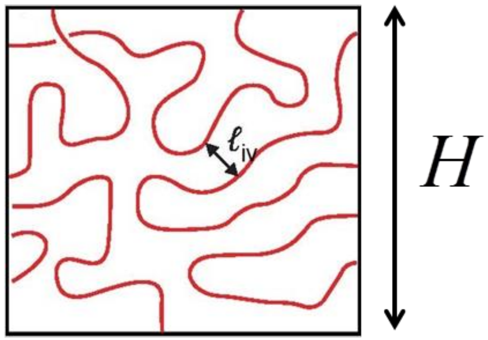
$$V_n = \frac{\dot{Q}}{A\,S\,T\,\rho} \qquad \mathcal{L} = \frac{6\,\pi\,s}{B\kappa}\left(\frac{A_0}{A} - 1\right)$$

- At distances of the order of the vortex lines core radius $R \sim a_0$ the dynamics of the vortex lines is described quantum mechanically by Gross-Pitaevskii (nonlinear Schrödinger) equation.

- At distances of the order of inter-vortex distance $a_0 \ll R \sim \ell \ll H$ the tangle dynamics may be described quasi-classically by Biot-Savart equation.

  Vortex Lines Density (VLD) $\mathcal{L}$

  inter-vortex distance $\ell = 1/\sqrt{\mathcal{L}}$



- At the macroscopic scale ($R \gg \ell$) the dynamics is described by a coupled two-fluid model (Hall-Vinen-Bekarevich- Khalatnikov equations) .

Self-consistent description of the superfluid channel flow requires the same level of description for normal and superfluid

"coarse-grained" Hall-Vinen-Bekarevich- Khalatnikov (HVBK) equations

$$\rho_n \frac{\partial \boldsymbol{V}_n}{\partial t} + \rho_n (\boldsymbol{V}_n \cdot \nabla) \boldsymbol{V}_n = -\frac{\rho_n}{\rho} \nabla p - \rho_s S \nabla T + \mathcal{F}_{ns} + \eta \Delta \boldsymbol{V}_n$$ NSE+ coupling

$$\rho_s \frac{\partial \boldsymbol{V}_s}{\partial t} + \rho_s (\boldsymbol{V}_s \cdot \nabla) \boldsymbol{V}_s = -\frac{\rho_s}{\rho} \nabla p + \rho_s S \nabla T - \mathcal{F}_{ns}$$ Euler Eq. + coupling

The coupling term $\mathcal{F}_{ns}$- mutual friction force

- accounts for the interaction between normal and superfluid
- depends on microscopic properties of the vortex tangle

On the macroscopic level, from dimensional reasoning $\mathcal{F}_{ns} \simeq \rho_s \kappa \, \alpha \, \mathcal{L} \, V_{ns}$

$\mathcal{L}$ - vortex tangle density

$V_{ns} = V_s - V_n$- relative velocity of normal and superfluid components

Need equation for VLD dynamics

Equation of Motion for VLD

$$\frac{d\,\mathcal{L}}{d\,t} = \underbrace{\mathcal{P}(t)}_{\text{production}} - \underbrace{\mathcal{D}(t)}_{\text{decay}}$$

Hall and Vinen works (1956-1958)

Vinen: Proc. R. Soc. A **238**, 204(1956)
Proc. R. Soc. A **242**, 493 (1957)
Proc. R. Soc. A **243**, 400 (1958)
Hall : Phil. Trans. A **250**, 359 (1957)

$$\frac{d\,\mathcal{L}}{d\,t} = \underbrace{\chi_1(T)\frac{B(T)}{2}\frac{\rho_n}{\rho}\mathcal{L}^{\frac{3}{2}}|V_{ns}|}_{\text{production}} - \underbrace{\chi_2(T)\frac{\hbar}{m}\mathcal{L}^2}_{\text{decay}}$$

Vinen equation

Main assumptions: isotropy, homogeneity, additivity

Often used in situation where basic assumptions are violated. Modifications are mostly limited to addition of more terms with different powers of $\mathcal{L}$.

Not the only form – the dimensional analysis dictate

$$\mathcal{P}(t) = \alpha\,\kappa\,\mathcal{L}^2 F(x), \quad D(t) = \alpha\,\kappa\,\mathcal{L}^2 G(x), \quad x = V_{ns}^2 \,/\, \kappa\,\mathcal{L}^2$$

The decay term $D(t) = \alpha\,\kappa\,\mathcal{L}^2 G(x)$ is V_{ns} independent, $G(x) = Const$

well supported by laboratory and numerical experiments

The production term $\mathcal{P}(t) = \alpha\,\kappa\,\mathcal{L}^2 F(x)$

allows different forms of $F(x)$ leading to different forms of $\mathcal{P}(t)$

Original Vinen's form $F(x) = \sqrt{x}$ $\quad \mathcal{P}_1(t) = \alpha\, C_1\, \mathcal{L}^{3/2} |V_{ns}|$

Modified Vinen's form $F(x) = x$ $\quad \mathcal{P}_2(t) = \alpha\, C_2\, \mathcal{L}\, {V_{ns}}^2/\kappa$

We suggest $F(x) = x^{3/2}$ $\quad \mathcal{P}_3(t) = \alpha\, C_3\, \sqrt{\mathcal{L}}\, {V_{ns}}^3/\kappa^2$

Can not be conclusively distinguished by time evolution of VLD in homogeneous flows.

Dynamics of VLD in inhomogeneous flow

$$\frac{d\mathcal{L}(\boldsymbol{r},t)}{dt} + \nabla\,\mathcal{J}(\boldsymbol{r},t) = \mathcal{P}(\boldsymbol{r},t) - \mathcal{D}(\boldsymbol{r},t)$$

VLD flux production decay

Specializing to the flow in a channel

$$\frac{d\mathcal{L}(y,t)}{dt} + \frac{d\mathcal{J}(y,t)}{dy} = \mathcal{P}(y,t) - \mathcal{D}(y,t)$$

$$\mathcal{P}_1(y,t) = \alpha\, C_1\, \mathcal{L}(y)^{3/2} |V_{ns}(y)|$$

$$\mathcal{P}_2(y,t) = \alpha\, C_2\, \mathcal{L}(y)\, {V_{ns}}^2(y)/\kappa$$

$$\mathcal{P}_3(y,t) = \alpha\, C_3\, \sqrt{\mathcal{L}}\, |V_{ns}|^3(y)/\kappa^2$$

$$D_{cl}(y,t) = \alpha\,\kappa\, C_d\, \mathcal{L}(y)^2$$

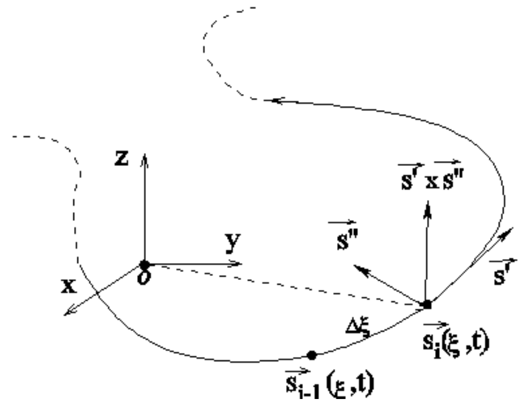
$$\mathcal{J}_{cl}(y,t) = -\frac{\alpha}{2\kappa}\frac{d{V_{ns}}^2(y)}{dy}$$

D. Khomenko, L. Kondaurova, V. S. L'vov, P. Mishra, A. Pomyalov, and I. Procaccia, Phys. Rev. B **91**, 180504(R) (2015)
D. Khomenko, V S. L'vov., A. Pomyalov., and I. Procaccia, in preparation.

Vortex Filament Method For Superfluid Dynamics

To test different forms of closure relations for $\mathcal{P}(y,t), \mathcal{D}(y,t)$ and $\mathcal{J}(y,t)$ we use numerical simulations of the channel flow in the framework of Vortex Filament Method (VFM).

K. W. Schwarz, Phys. Rev. B, 38, 2398 (1988)



$$\frac{d\boldsymbol{s}(\xi,t)}{dt} = \boldsymbol{V}^{\rm s}(\boldsymbol{s},t) + \left(\alpha - \alpha'\boldsymbol{s}'\times\right)\mathbf{s}'\times\boldsymbol{V}_{\rm ns}(\boldsymbol{s},t)$$

$$\boldsymbol{V}^{\rm s}(\boldsymbol{s},t) = \boldsymbol{V}_0^{\rm s} + \boldsymbol{V}_{\rm BS}(\boldsymbol{s})$$ Superfluid velocity

applied velocity ($\boldsymbol{V}_0^{\rm s}$)   Biot-Savart velocity ($\boldsymbol{V}_{\rm BS}$)

Vortex line is parameterized by a directional curve $\boldsymbol{s}(\xi,t)$

$\boldsymbol{s}''$ ~local curvature

$\boldsymbol{s}'\times\boldsymbol{s}''$ ~local velocity

$$\boldsymbol{V}_{\rm BS}(\boldsymbol{s}) = \frac{\kappa}{4\pi}\int_{\mathcal{C}}\frac{(\boldsymbol{s}-\boldsymbol{s}_1)\times\boldsymbol{d}s_1}{|\boldsymbol{s}-\boldsymbol{s}_1|^3} \Rightarrow \boldsymbol{V}_{\rm LIA}^{\rm s} + \boldsymbol{V}_{\rm nl}^{\rm s}(\boldsymbol{s})$$

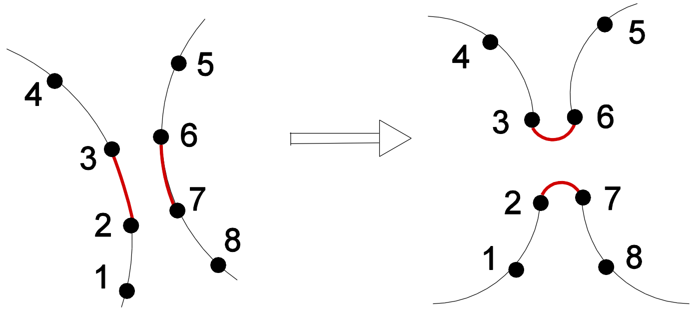
$$\boldsymbol{V}_{\rm LIA}^{\rm s}(\boldsymbol{s}) = \beta\boldsymbol{s}'\times\boldsymbol{s}'' \qquad \beta = \frac{\kappa}{4\pi}\ln\frac{c\,R}{a_0}$$

$$\boldsymbol{V}_{\rm nl}^{\rm s}(\boldsymbol{s}) = \frac{\kappa}{4\pi}\int_{\mathcal{C}'}\frac{(\boldsymbol{s}-\boldsymbol{s}_1)\times\boldsymbol{d}s_1}{|\boldsymbol{s}-\boldsymbol{s}_1|^3}$$

$$\boldsymbol{V}_{ns}^0 = \boldsymbol{V}^n - \boldsymbol{V}_0^s - \boldsymbol{V}_{nl}^s$$

Unlike classical vortices, quantum vortex lines stretch and reconnect
without changing the structure and the size of the core
Not described by equations on BS level ⟹ In VFM introduced by artificial procedures

Reconnections

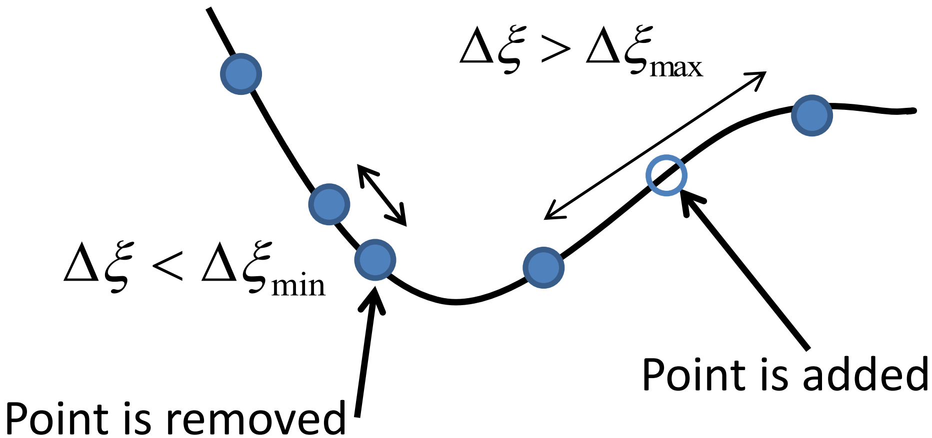


- The energy of the line is proportional to the length
- Reconnection is a dissipative process

Reconnect lines if
$$\begin{cases} \ell_{2-7} < \Delta\xi \\ \ell_{2-7} + \ell_{3-6} < \ell_{2-3} + \ell_{6-7} \end{cases}$$

Dynamic re-meshing

During evolution each points moves with its own velocity- the distance between points changes. To maintain accuracy of calculations , points are added or removed from line at each time step.



On the microscopic level the dynamical balance between vortex-line growths and decay is defined by an instantaneous rate of change of a line element of length $\delta\xi$

$$d\delta\xi/(\delta\xi dt) = \boldsymbol{s}' \cdot d\boldsymbol{s}'/dt = \alpha \boldsymbol{V}_{\rm ns} \cdot (\boldsymbol{s}' \times \boldsymbol{s}'') + \boldsymbol{s}' \cdot \boldsymbol{V}_{\rm nl}^{\rm s\,\prime} - \alpha' \boldsymbol{s}'' \cdot \boldsymbol{V}_{\rm ns}$$

giving after a proper integration the closed set of equations for VLD dynamics

$$\frac{\partial\, \mathcal{L}(y,t)}{\partial t} + \frac{\partial\, J(y,t)}{\partial y} = \mathcal{P}(y,t) - D(y,t)$$

$$D(y,t) = \frac{\alpha\beta}{\Omega} \int d\xi\, |s''|^2$$

$$\mathcal{J}(y,t) = \frac{\alpha}{\Omega} \int d\xi\, \boldsymbol{V}_{drift} = \frac{\alpha}{\Omega} \int d\xi (\boldsymbol{V}^s + \alpha\, \boldsymbol{s}' \times \boldsymbol{V}_{ns})$$

$$\mathcal{P}(y,t) = \frac{\alpha}{\Omega} \int d\xi\, (\boldsymbol{V}^n - \boldsymbol{V}_0^s - \boldsymbol{V}_{nl}^s) \cdot (\boldsymbol{s}' \times \boldsymbol{s}'')$$

The closure relations in terms of vortex tangle properties

Decay term

$$D(y,t) = \frac{\alpha\beta}{\Omega}\int d\xi \; |s"|^2 = \alpha\beta\mathcal{L}\,\tilde{S}^2, \qquad \tilde{S}^2 = c_2^2\mathcal{L}$$

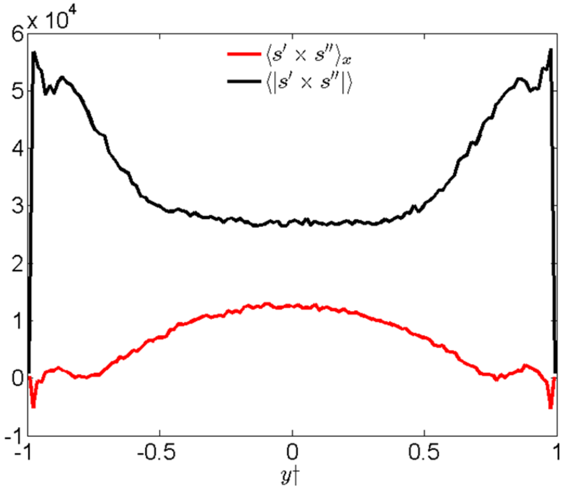
$$D_{cl}(y,t) = \alpha\,\kappa\,C_d\mathcal{L}^2 \qquad C_d = \frac{\beta c_2^2}{\kappa}$$

Flux toward the wall

$$\mathcal{J}(y,t) = \frac{\alpha}{\Omega}\int d\xi \; V_{drift} = \frac{\alpha}{\Omega}\int d\xi \; V_{ns,x}s'_z$$

$$\mathcal{J}_{cl}(y,t) = -\frac{\alpha}{\kappa}V_{ns}\frac{dV_s(y)}{dy}$$

Production term



$$\mathcal{P}(y,t) = \frac{\alpha}{\Omega}\int d\xi \boldsymbol{V}_{ns}^0 \cdot (\boldsymbol{s}' \times \boldsymbol{s}")$$

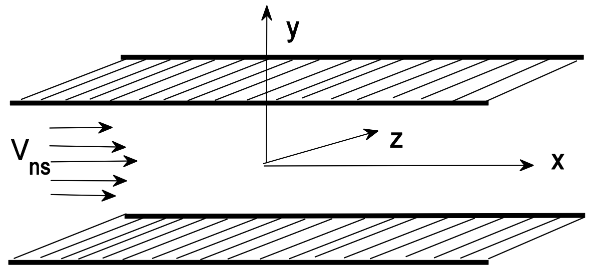
$$\mathcal{P}(y,t) = \alpha\,\mathcal{L}\boldsymbol{V}_{ns}^0 \cdot \langle \boldsymbol{s}' \times \boldsymbol{s}" \rangle$$ How to model ???

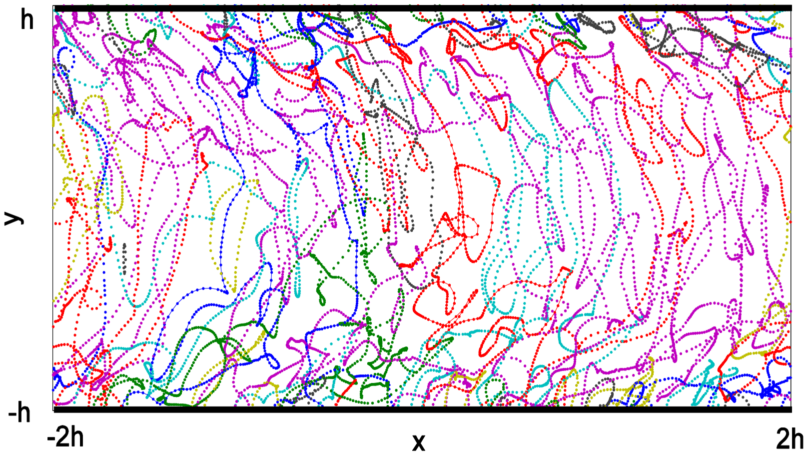
$$\boldsymbol{V}_{ns}^0 \cdot \langle \boldsymbol{s}' \times \boldsymbol{s}" \rangle \sim V_{ns} |\, \boldsymbol{s}' \times \boldsymbol{s}" | \Longrightarrow \mathcal{P}_1(y,t)$$

$$\boldsymbol{V}_{ns}^0 \cdot \langle \boldsymbol{s}' \times \boldsymbol{s}" \rangle \sim V_{ns}\,V_{loc,x} \Longrightarrow \mathcal{P}_3(y,t)$$

Numerical Setup

We consider counterflow in a planar channel

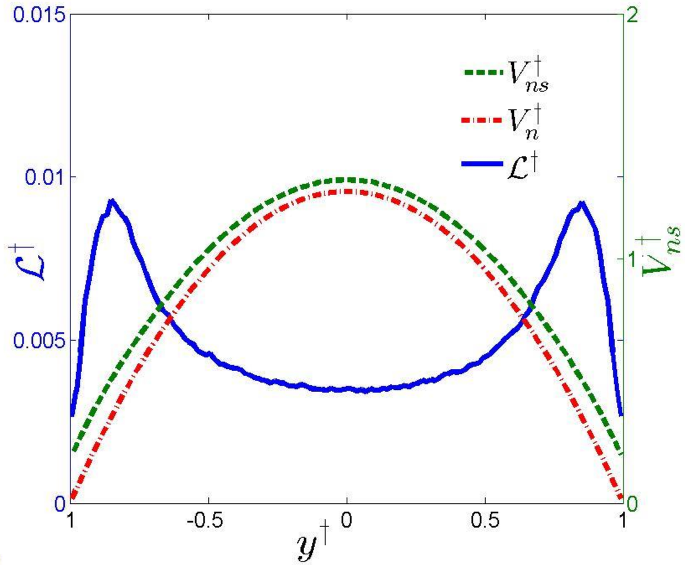




- Full Biot-Savart calculations
- Computational domain 0.2x0.1x0.1 cm
- Periodic boundary conditions in x, z directions
  Solid walls with slip conditions in y direction
- Line resolution $\Delta\xi = 1.6 \times 10^{-3}$ cm
- Dissipative reconnection criterion
- T=1.6 K, $\alpha = 0.098, \alpha' = 0.016$, ρ_s/ρ_n=5.17.
- V_0^s calculated dynamically from the zero net mass flux condition
  $$\rho_n \langle \boldsymbol{V}_n \rangle = \rho_s \langle V_0^s + \boldsymbol{V}_{BS} \rangle$$
- Three normal velocity profile types

Results

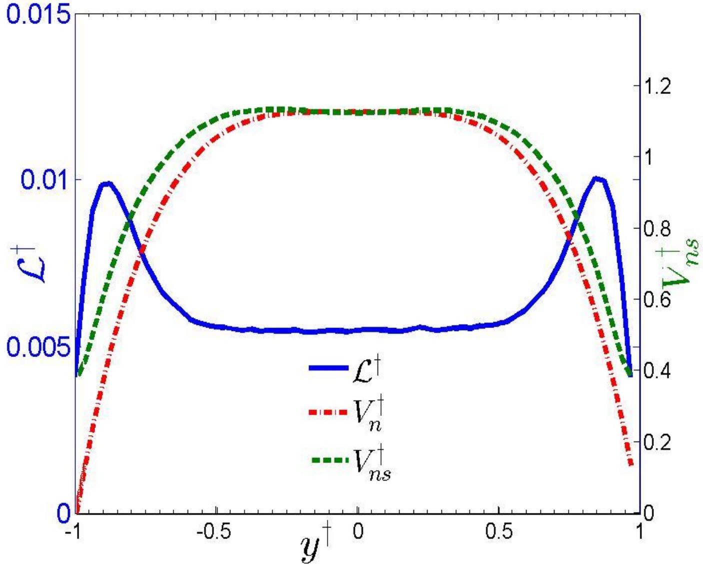
VLD and V_n

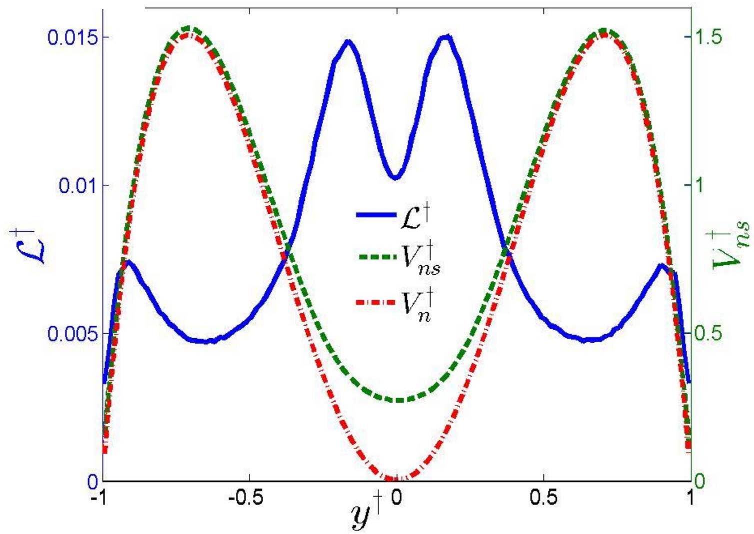


Parabolic profile

Flat-top profile

Non-Parabolic profile





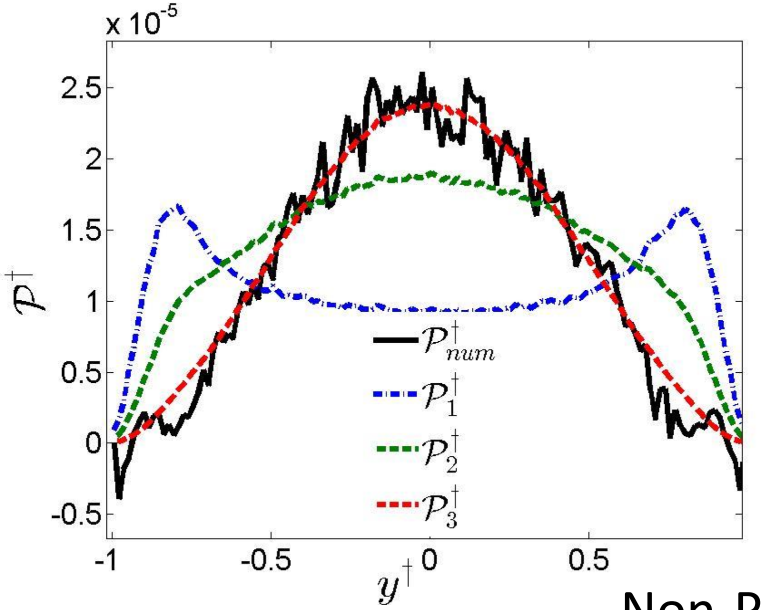
Normalized variables: $y^{+} \equiv y/h$; $\mathcal{L}^{+} \equiv \kappa^2 \mathcal{L}/\langle V_{ns}^2 \rangle$; $V^{+} \equiv V/\sqrt{\langle V_{ns}^2 \rangle}$

Production

$\mathcal{P}_1(y,t) \propto \mathcal{L}^{3/2}|V_{ns}|$ Original Vinen's

$\mathcal{P}_2(y,t) \propto \mathcal{L}\,|V_{ns}|^2$ Modified Vinen's

$\mathcal{P}_3(y,t) \propto \sqrt{\mathcal{L}}\,|V_{ns}|^3$ New form



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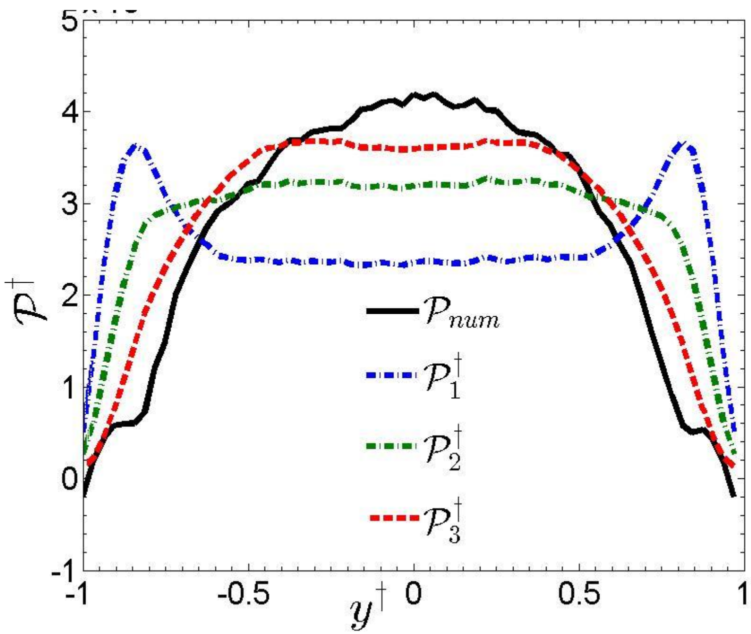
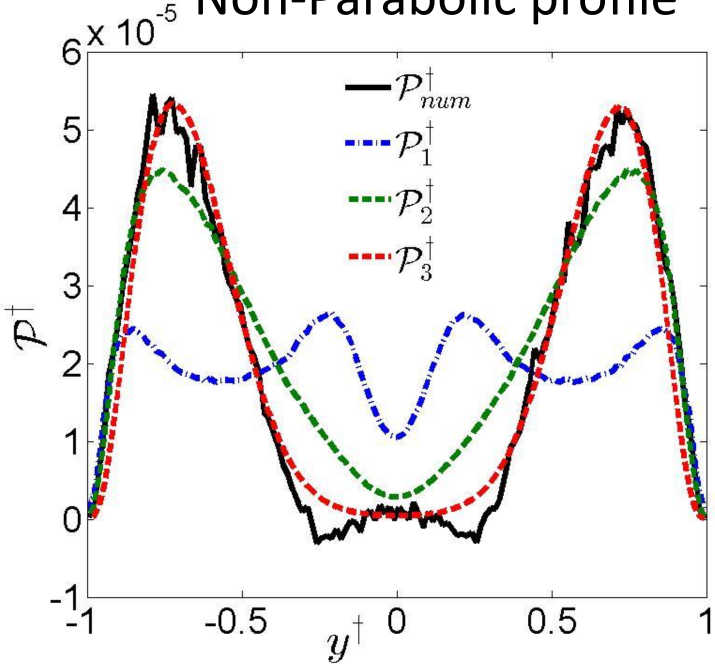


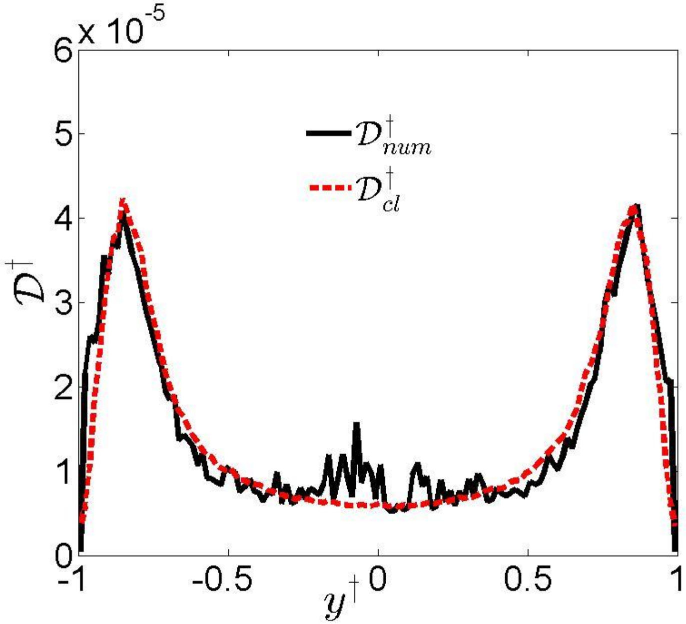
Non-Parabolic profile



Normalized variables: $y^+ \equiv y/h$; $\mathcal{P}^+ \equiv \kappa^3 \mathcal{P}/\langle V_{ns}^2\rangle^2$

Decay

$$D_{cl}(y,t) = \alpha\,\kappa\,C_d\mathcal{L}^2$$



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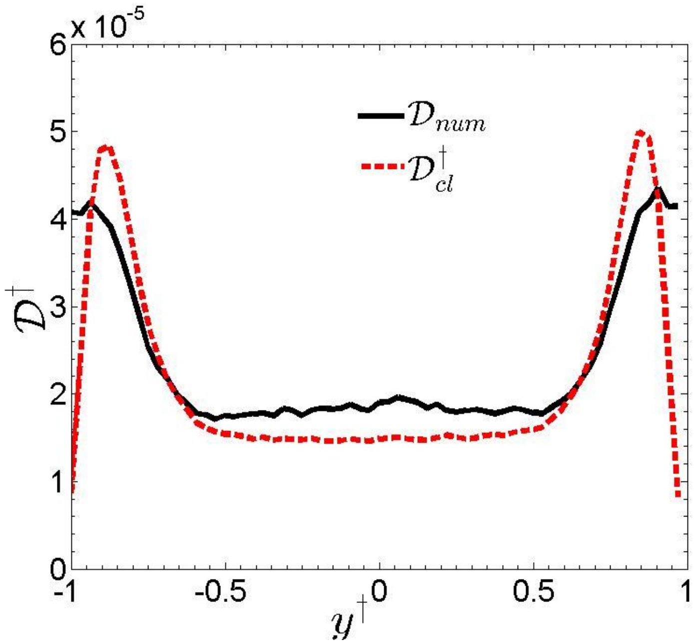
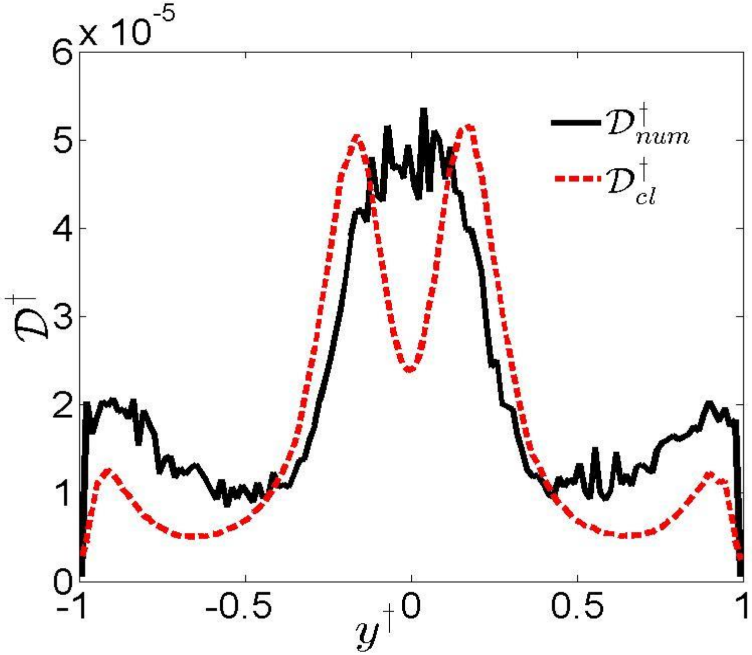




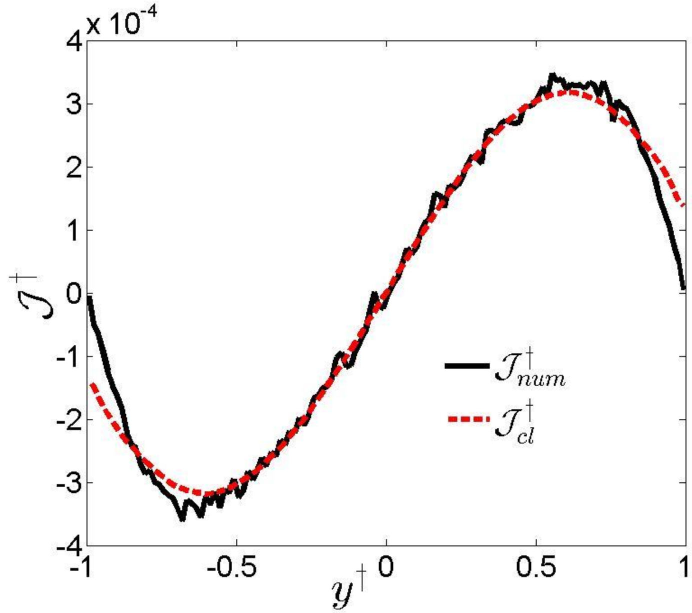
Normalized variables: $y^{+} \equiv y/h$; $\mathcal{D}^{+} \equiv \kappa^3\mathcal{D}/\langle V_{ns}^2\rangle^2$

Flux of VLD

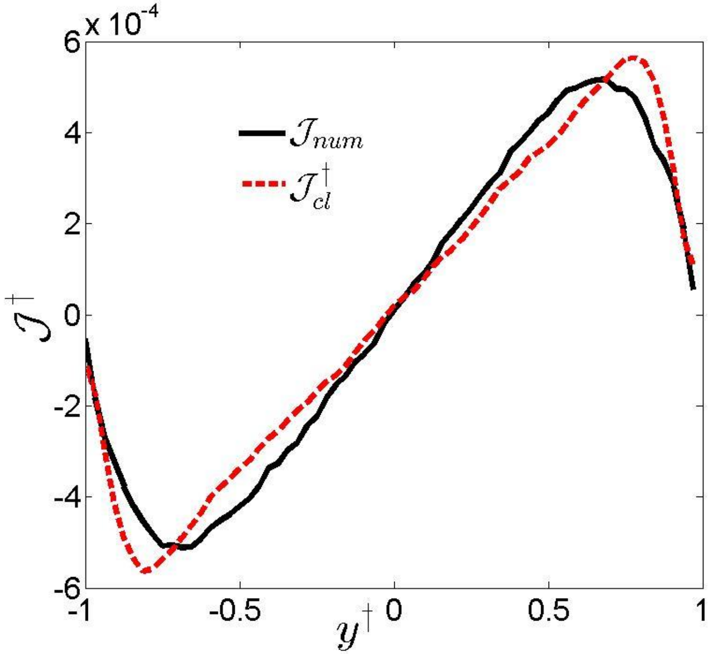
$$\mathcal{J}_{cl}(y,t) = -\frac{\alpha}{\kappa}\boldsymbol{V}_{ns}^{0}\,\frac{dV_s(y)}{dy}$$

No fitting parameters!

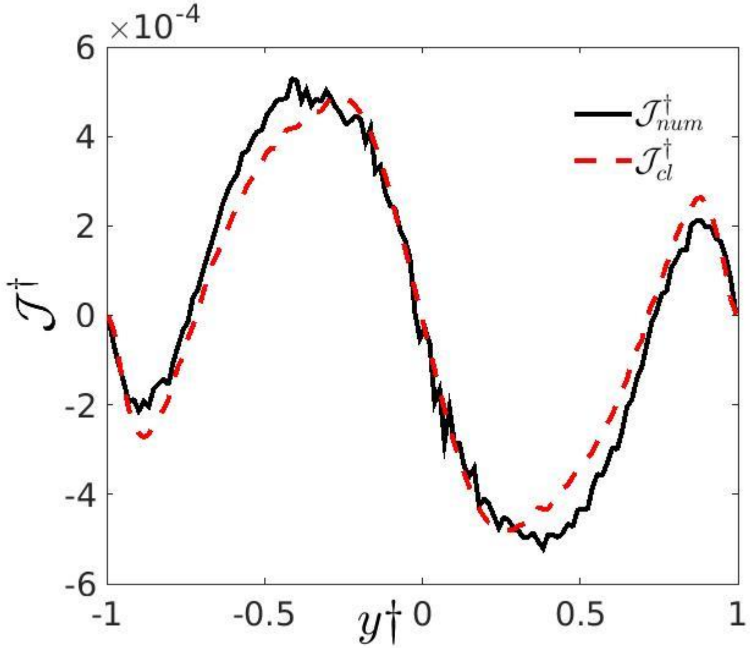
Parabolic profile



Flat-top profile



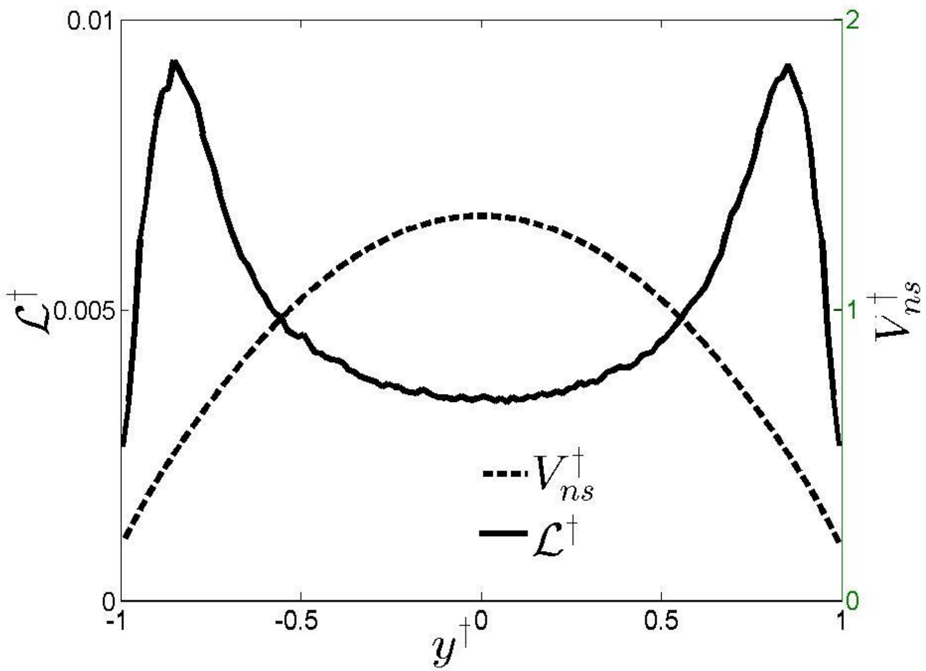
Non-Parabolic profile

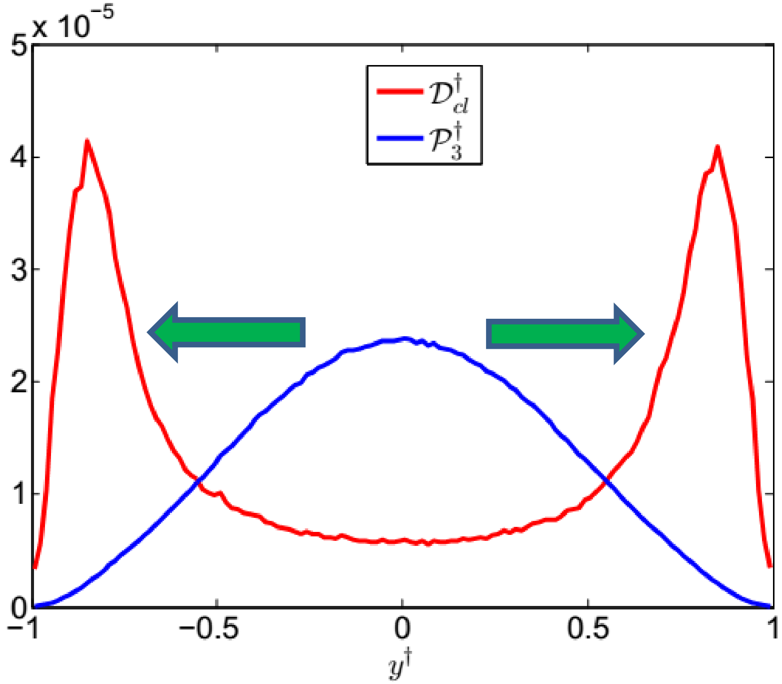


Normalized variables: $y^{+} \equiv y/h$; $\mathcal{J}^{+} \equiv \kappa^2 \mathcal{J}/\langle V_{ns}^2\rangle^{3/2}$

How it works?

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{\alpha}{\kappa}\frac{\partial}{\partial y}\left[V_{ns}\frac{dV_s}{dy}\right] = \frac{\alpha C_3}{\kappa^2}\sqrt{\mathcal{L}}\;|V_{ns}|^3 - \alpha\,\kappa\,C_d\mathcal{L}^2$$





Summary

- We have suggested the equation of motion for the vortex tangle line density in the inhomogeneous flows with closure relations for the production, decay and VLD flux terms via $\mathcal{L}$ and V_{ns} only.

- We have verified the proposed closures by direct numerical simulation using VFM in a plane channel.

- We found quantitative agreement between the proposed closures and the numerical results for different types of the normal velocity component profiles.